Eur. Phys. J. B **48**, 71–73 (2005)
DOI: 10.1140/epjb/e2005-00373-2

THE EUROPEAN PHYSICAL JOURNAL B

Random phases in Bose-Einstein condensates with higher order nonlinearities

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Received 25 May 2005 / Received in final form 17 July 2005
Published online 9 December 2005 –

Abstract. We present a statistical description of Bose-Einstein condensates with general higher order nonlinearities. In particular, we investigate the case of cubic-quintic nonlinearities, of particular interest for dilute condensates. The implication of decoherence for the stability properties of the condensate is discussed.

PACS. 03.75.Lm Tunneling, Josephson effect, Bose-Einstein condensates in periodic potentials, solitons, vortices and topological excitations – 05.45.-a Nonlinear dynamics and nonlinear dynamical systems – 67.40.Vs Vortices and turbulence – 67.57.De Superflow and hydrodynamics

The basic concept of macroscopic quantum states, such as Bose–Einstein condensates (BECs) [1] and lately also Fermi condensates [2], has caught the interest of the physics community, both due to the nature of the concept itself but also since the possibilities to perform new and exciting experiments was early recognized [1]. BECs are normally described by the Gross-Pitaevskii (GP) equation [3], in which the cubic nonlinearity represents two-body forces between the bosons in the condensate. There are numerous works on the theoretical foundations and implications of the GP equation [4–6]. When the scattering length is positive, the possibility of dark solitons is given via the GP equation. Such dark solitons are stable towards perturbations in one dimension, while the multi-dimensional case is more complex. Such dark solitons have also been found experimentally [7]. In the case of a negative effective scattering length, the GP equation admits bright solitons, which are prone to collapse in dimensions larger than one. Such bright solitons have also been experimentally obtained, both in terms of trains of solitons [8], as well as single solitary structures [9]. The dynamics of these bright soliton structures has also been analyzed both analytically [10] and numerically [11].

If three-body interactions are taken into account, higher order nonlinearities will modify the GP equation. We may write this as a generalized NLSE of the form [12]

$$i\hbar\partial_t\psi + \frac{\hbar^2}{2m}\nabla^2\psi + \alpha|\psi|^2\psi + \beta|\psi|^4\psi = 0, \qquad (1)$$

where α and β in general is complex-valued, and ψ is the condensate wave function. As β goes to zero, we regain the GP equation. The real parts of α and β corresponds to elastic collisions within the condensates, while the imaginary part appears due to inelastic scattering [13]. In what follows we will neglect the collisional losses, and assume that α and β is real, and can take on both positive and negative values. In fact, the coefficient α is proportional to a, the scattering length, which can be tuned to take on both negative and positive values (see, e.g. Refs. [14] and [17]), while β is proportional to a^4 for dilute systems [15–17]. Thus, in BECs where a is large, the quintic contribution to equation (1) may become significant. Equation (1) can be demonstrated to have solitary solutions in one and two dimensions [12], and appears not only in the physics of BECs, but also in, e.g. nonlinear optics [18]. Furthermore, equation (1) is a special case of the equation

$$i\hbar\partial_t\psi + \frac{\hbar^2}{2m}\nabla^2\psi + U\left(|\psi|^2\right)\psi = 0, \qquad (2)$$

where U in general is a complex-valued function of the norm of the wave function squared.

The stability of solutions to equation (1) towards coherent perturbations was analyzed by [19], where the growth rate for the modulational instability was found. This growth rate signifies the onset of purely growing perturbations, and is thus an important indication of the possibility of solitary solutions. However, the effects of incoherence, e.g. a random phase in the wave function, may significantly alter the modulational instability and therefore also the onset of inhomogeneity growth, and is an important issue (for a discussion, see, e.g. Refs. [20] and [21]). A very direct approach in analyzing the effects of partial coherence lies in the Wigner formalism [22,23]. This approach has found uses in the study of surface gravity

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waves in fluids [24] and electromagnetic waves in nonlinear media [25], in quantum statistical mechanics [26], in nonlinear optics [27], and in quantum plasmas [28]. Here we will apply the Wigner formalism to the problem of partial coherence in the modulational instability of higher order nonlinear BECs.

In order to analyze the statistical properties of equation (1), we may introduce the Fourier transform of the two-point correlation function of the wave function, i.e. the Wigner function, according to

$$F(t,\mathbf{r},\mathbf{p})=\frac{1}{(2\pi\hbar)^s}\int d\xi\, e^{i\mathbf{p}\cdot\xi/\hbar}\langle\psi^*(\mathbf{r}+\xi/2,t)\psi(\mathbf{r}-\xi/2,t)\rangle \tag{3}$$

for the wave function ψ, where s denotes the dimensionality of the problem at hand, the asterisk is the complex conjugate operation, and the angular bracket denotes the ensemble average. The Wigner function corresponds to a generalized distribution function for the bosons, and by applying the time derivative to equation (3) and using equation (2), one finds the Vlasov-like equation

$$\partial_t F + \frac{1}{m}\mathbf{p}\cdot\nabla F + \frac{2}{\hbar}U(|\psi|^2)\sin\left(\frac{\hbar}{2}\overleftarrow{\nabla}\cdot\overrightarrow{\nabla}_p\right)F = 0. \tag{4}$$

where the sin-operator is defined in terms of its Taylor expansion and the arrows denote the direction of operation. In the case of of a cubic-quintic nonlinearity, such as in equation (1), we obtain

$$\partial_t F + \frac{1}{m}\mathbf{p}\cdot\nabla F + \frac{2}{\hbar}(\alpha|\psi|^2+\beta|\psi|^4)\sin\left(\frac{\hbar}{2}\overleftarrow{\nabla}\cdot\overrightarrow{\nabla}_p\right)F = 0 \tag{5}$$

Moreover, the modulus square of the wave function is given by

$$|\psi|^2 = \int d\mathbf{p}\, F(t,\mathbf{r},\mathbf{p}). \tag{6}$$

For the sake of clarity, we now focus on the one-dimensional case. The stability of equation (4) can be analyzed using a linearization procedure. Letting $F = F_0(p) + f(p)\exp(ikz - i\omega t)$, where $|f| \ll F_0$, we linearize equation (4) in order to obtain the nonlinear dispersion relation

$$1 = -\frac{m}{\hbar k}\frac{dU}{d|\psi_0|^2}\int dp\, \frac{F_0(p+\hbar k/2) - F_0(p-\hbar k/2)}{p - \omega m/k}. \tag{7}$$

Equation (7) is the general dispersion relation for matter waves taking into account higher order nonlinearities.

For the case of a cubic-quintic nonlinearity, equation (7) reduces to

$$1=-\frac{m}{\hbar k}\Big(\alpha+2\beta|\psi_0|^2\Big)\int dp\, \frac{F_0(p+\hbar k/2) - F_0(p-\hbar k/2)}{p - \omega m/k}. \tag{8}$$

For the monochromatic wave case, i.e. $F_0(p) = |\psi_0|^2\delta(p - p_0)$, we obtain the dispersion relation [19]

$$\omega = \frac{p_0 k}{m} \pm \left[\frac{\hbar^2 k^4}{4m^2} - \frac{k^2|\psi_0|^2}{m}\Big(\alpha + 2\beta|\psi_0|^2\Big)\right]^{1/2} \tag{9}$$

from equation (8). Setting $\beta = 0$ in the dispersion relation (9), we obtain the Bogolubov expression [29] for the elementary excitations of the BEC. The standard method employed in obtaining the result (9) with $\beta = 0$ is to set $\psi = \psi_0 + \psi_1$, where $|\psi_1| \ll |\psi_0|$ and ψ_0 is the background state, and linearizing equation (1), after which the equation may be split into its real and imaginary part and harmonically decomposed. We note that the Wigner approach presented in this paper is equivalent in to the Bogolubov method in the monochromatic limit. Letting $\omega = p_0k/m + i\gamma$ in the above equation, we obtain the modulational instability growth rate [19]

$$\gamma = \left[\frac{k^2|\psi_0|^2}{m}(\alpha + 2\beta|\psi_0|^2) - \frac{\hbar^2 k^4}{4m^2}\right]^{1/2}. \tag{10}$$

If the waves are not exactly monochromatic, but have a spectral broadening due to, e.g. a random phase in the background wave function, we may model equilibrium condensate spectrum by a Lorentzian distribution [30]

$$F_0(p) = \frac{|\psi_0|^2}{\pi}\frac{p_T}{(p-p_0)^2 + p_T^2}, \tag{11}$$

where p_T denotes the width of the distribution. The Lorentzian distribution solves equation (5), as well as the more general (4), and is thus a valid perturbation background. We note that the phase fluctuations may stem from a variety of perturbations, e.g. thermal effects or quantum fluctuations [21]. The dispersion relation (7) for this case then turns out to be

$$\omega = \frac{p_0 k}{m} \pm \left[\frac{\hbar^2 k^4}{4m^2} - \frac{k^2}{m}\frac{dU}{d|\psi_0|^2}|\psi_0|^2\right]^{1/2} - i\frac{p_T k}{m}, \tag{12}$$

which gives the growth rate

$$\gamma = \left[\frac{k^2}{m}\frac{dU}{d|\psi_0|^2}|\psi_0|^2 - \frac{\hbar^2 k^4}{4m^2}\right]^{1/2} - \frac{p_T k}{m}. \tag{13}$$

The growth rate (13) is valid for a general nonlinearity. We note that if $dU/d|\psi_0|^2 \le 0$, there is no modulational instability growth, and the perturbations are damped. Thus, a minimum requirement for a positive growth rate for an arbitrary nonlinearity in equation (2) is that $dU/d|\psi_0|^2 > 0$.

In the case of a cubic-quintic nonlinearity $U(|\psi|^2) = |\psi|^2(\alpha + \beta|\psi|^2)$, equation (12) reduces to

$$\omega = \frac{p_0 k}{m} \pm \left[\frac{\hbar^2 k^4}{4m^2} - \frac{k^2|\psi_0|^2}{m}\Big(\alpha + 2\beta|\psi_0|^2\Big)\right]^{1/2} - i\frac{p_T k}{m}, \tag{14}$$

which gives a purely growing modulational instability whose growth rate is

$$\gamma = \left[\frac{k^2|\psi_0|^2}{m}\left(\alpha + 2\beta|\psi_0|^2\right) - \frac{\hbar^2 k^4}{4m^2}\right]^{1/2} - \frac{p_T k}{m}. \tag{15}$$

Comparing with equation (15) with equation (10), one clearly sees the damping character of the spectral broadening term. We note that since α and β may be positive or negative, independent of each other, the instability properties crucially depends on the nonlinear terms in the expression (13). If $\alpha, \beta < 0$ (the defocusing case), modulational instability growth is not possible, and this corresponds to the well-known stability of dark solitary solutions. However, if we have $\alpha < 0$ (defocusing cubic nonlinearity), while $\beta > 0$ (focusing quintic nonlinearity), we may have a new instability regions, not present in the Gross–Pitaevskii equation. If $\alpha > 0$ (focusing cubic nonlinearity) and $\beta < 0$ (defocusing quintic nonlinearity) we will have a damping in the growth rate due to three-body interaction. Note that this damping is quite different from the dissipation due to inelastic three-body scattering, and has more the character of Landau damping [21]. It is also clear that the case $\alpha, \beta > 0$ (the focusing case) gives the maximum instability growth rate. However, even in this case, a very broad spectral distribution of the BEC, due to e.g. thermal noise, may quench the growth rate considerably, even removing it all together. We would like to stress that the Wigner method presented in this paper is a very general approach to partial coherence and spectral broadening. Thus, the method is appropriate for a variety of spectral distributions, e.g. Gaussians, as well as the Lorentizian (11), but this will be pursued in future research.

To summarize, we have presented a perturbation analysis of the statistical properties of the generalized Gross–Pitaevskii equation (1), by using the Wigner formalism. In the case of a random phase of the background wave function of the condensate, we find that the spectral broadening gives rise to a reduced growth rate, as compared to the mono-chromatic case.

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